Crossing Relations and the Prediction of Symmetries in a Soluble Model*

A. W. MARTIN† AND W. D. MCGLINN
Argonne National Laboratory, Argonne, Illinois
(Received 28 July 1964)

The two-channel static model, incorporating unitarity and a nontrivial crossing relation, is known to be exactly soluble for a specific choice of the crossing matrix. The solution, obtained by Wilson and independently by Wanders, is largely independent of a number of the dynamical assumptions of the static model and may, therefore, indicate properties of more realistic models for the strong interactions. We have generalized the two-channel problem by introducing arbitrary crossing matrices of a type suitable to the static model. We ask if the requirement that exact solutions exist leads to restrictions upon the crossing matrices and hence to the prediction of symmetries. The answer is in the negative; we exhibit solutions for all values of the parametrized crossing matrix. However, special solutions of a particularly simple form exist only for crossing matrices corresponding to the symmetry group SU(2). The relationship between these special solutions and the general solution is discussed.

I. INTRODUCTION

ONE of the more intriguing ideas to emerge from the study of the strong interactions is that the internal symmetries possessed by those interactions may be consequences of the dynamical laws of motion. That is, isotopic spin or the higher symmetry of the eightfold way[1] are not symmetries to be imposed *ab initio* (symmetries whose origin we can never understand), but rather they are symmetries dictated by the requirement that there exist solutions to the dynamical equations. For the most part, the development of this idea has followed the "bootstrap" philosophy; it is the requirement of self-consistency among the particles, in the sense that no particles are fundamental and independent of the existence of the others, that leads to the symmetries exhibited by the strongly interacting particles.

Applications of the bootstrap hypothesis to the prediction of symmetries can be roughly catalogued as "algebraic" or "dynamic." The algebraic approach, as typified by the work of Cutkosky,[2] and Sudarshan,[3] avoids restriction as to the number or the characteristics of the particles and derives the underlying group algebra through the requirement that all the particles be on equal footing. In this treatment the dynamics of the strong interactions is often suppressed. In the dynamic approach, as typified by the work of Capps,[4] the emphasis is placed upon the possibility of a self-consistent solution of the dynamical equations for a given number of particles with specified characteristics (isospin, hypercharge, etc.)

While both of these approaches appear fruitful, neither one fully incorporates two of the most fundamental features of the strong interactions, the crossing relations and the requirement of unitarity. Wigner[5] has emphasized the important role of the crossing relations in providing a transition between the geometrical symmetry laws and the symmetries of the strong interactions. He notes that the crossing relations do not refer to any particular type of interaction and that their validity is in general unquestioned, both of these being primary properties of the geometrical symmetry laws.

The purpose of this paper is to examine to what extent the crossing relations and unitarity can be expected to lead to the prediction of symmetries in the strong interactions. Of course, it has long been recognized[6] that the successful combination of crossing symmetry and unitarity is perhaps the most basic problem in strong-interaction physics, and we will lean heavily on the work of others in this analysis. There is yet another aspect of the bootstrap hypothesis which must be discussed before the analysis can proceed. It has been suggested[7] that the full solution of the dynamical equations of motion may comprise no more than an identity; that it is the approximations employed rather than the equations themselves that seem to indicate the existence of unique solutions in the bootstrap sense.

While it seems impossible at this time to answer this question one way or the other, there is an important moral here for the present considerations. If we are to predict symmetries of the strong interactions on the basis of the crossing relations and unitarity alone, then we must avoid the possibility that the symmetries are a consequence of approximations made

* Work performed under the auspices of the U. S. Atomic Energy Commission.
† Present address: Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California.

[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).

[2] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963); R. E. Cutkosky and M. Leon (to be published).

[3] E. C. G. Sudarshan, Phys. Letters **9**, 286 (1964); E. C. G. Sudarshan, L. O'Raifeartaigh, and T. S. Santhanam (to be published).

[4] R. H. Capps, Phys. Rev. Letters **10**, 312 (1963); Nuovo Cimento **30**, 340 (1963); Phys. Rev. **134**, B460 (1964).

[5] E. P. Wigner, Phys. Today **17**, 34 (1964).

[6] See, for example, G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1962).

[7] B. Diu and H. R. Rubinstein, Phys. Letters **8**, 203 (1964). See also M. Gell-Mann and F. Zachariasen, *ibid.* **10**, 129 (1964).

in the course of the analysis. The framework for our considerations may now be stated.

We will examine an exactly soluble model which incorporates the unitarity requirement and a nontrivial crossing symmetry. In fact, because the symmetries of the strong interactions can be expressed in terms of the crossing relations, we must work with a model that allows a general formulation of the crossing requirement. Finally, we wish to suppress as far as possible any dependence of the model on specific dynamical assumptions. The stringent requirement of an exactly soluble model does not, however, permit us to eliminate dynamical assumptions entirely, and as a result the analysis is limited to a special, although large, class of crossing relations.

The model we consider is a generalization of the two-channel static model solved by Wilson[8] and independently by Wanders.[9] These authors obtained the general solution of the unitarity and crossing requirements, within the framework of the static model, for the crossing matrix corresponding to the scattering of isospin-1 particles by isospin-$\frac{1}{2}$ particles (as one of the possible interpretations). This specific crossing matrix can be replaced by a general two-by-two crossing matrix, expressible in terms of a single continuous parameter, and the analogous solution procedure can be followed.

The question we ask is whether solutions exist for this problem for all values of the parameter or only for specific values. That is, does the requirement of an exact solution of unitarity and crossing restrict the allowed crossing matrices to a special subset of the entire class? And if so, to what symmetry does this subset of crossing matrices correspond? It will become clear that this approach to the question of the prediction of symmetries has little, if anything, to do with the bootstrap hypothesis. The requirements of self-consistency and equal footing among the strongly interacting particles are replaced by the more general requirements of unitarity and crossing symmetry.

Section II contains the mathematical statement of the problem and the special solutions obtained by generalizations of Wanders' procedure. The general solution is derived in Sec. III by means of a technique developed by Albright and McGlinn.[10] The relationship between the special and the general solutions is discussed in Sec. IV. Appendix I deals with the properties of the special class of crossing matrices under consideration, while Appendix II contains a derivation of the pertinent subset of crossing matrices.

II. SPECIAL SOLUTIONS

The analysis of this section is based almost entirely upon the mathematical procedure presented by Wanders,[9] and we will follow his notation in order to facilitate comparison between the two developments. The structure of the problem to be solved is this: We wish to find two elements of the scattering matrix, $S_\alpha(z)$, $\alpha=1, 2$, functions of a single (energy) variable, that are uncoupled from each other in the physical region $z\geqslant 1$. That is,

$$S_\alpha(z+i\epsilon)=\exp[2i\delta_\alpha(z)],\quad \delta_\alpha(z) \text{ real} \quad \text{for} \quad z\geqslant 1. \quad (1)$$

These S-matrix elements are assumed to be coupled in unphysical region, however, by virtue of the crossing relations. These statements, combined with the assumption of analyticity in the complex z plane, define the problem.

The mathematical statement of the required properties of the S-matrix elements is: (i) analyticity—$S_\alpha(z)$ is meromorphic in the z plane with cuts along $(-\infty, -1)$ and $(1, \infty)$; $S_\alpha(z^*)=S_\alpha{}^*(z)$, (ii) crossing symmetry—

$$S_\alpha(-z)=\textstyle\sum_\beta A_{\alpha\beta}S_\beta(z)\,, \quad (2)$$

and (iii) unitarity—$S_\alpha(z)$ has only one branch point on the positive real axis; it is at $z=1$ and is of the type $(z-1)^{1/2}$. The continuation $S_\alpha{}^{(2)}(z)$ on the second sheet of this branch point is given by

$$S_\alpha{}^{(2)}(z)=1/S_\alpha(z)\,. \quad (3)$$

These conditions are the conditions of the static model in the one-meson approximation (elastic unitarity). The statements of analyticity and unitarity, including the nature of the physical branch point, however, are generally accepted features of more realistic models for the strong interactions. In this regard, the primary limitation imposed by the static model is the structure of the crossing relation [Eq. (2)]. Other dynamical features of the static model, such as the location of poles and the necessity for cutoffs, are immaterial to the present considerations.

The special class of crossing matrices appropriate to the static model is discussed in Appendix I. It is shown there that each crossing matrix must have the properties: (i) It is equal to its own inverse, and (ii) the sum over each row of the matrix equals unity. The general form of a two-by-two matrix enjoying these properties is readily found to be

$$A=\begin{pmatrix} c & 1-c \\ 1+c & -c \end{pmatrix}, \quad (4)$$

where c is an arbitrary constant. The real analyticity of the S-matrix elements restricts c to be real. We now propose to follow Wanders' procedure with the general crossing matrix of Eq. (4) and to obtain exact solutions for the scattering matrix elements $S_\alpha(z)$.

The first step is the solution of the crossing relation. This is a simple algebraic problem with the result

$$\begin{aligned} S_1(z)&=s(z)-(1-c)a(z)\,,\\ S_2(z)&=s(z)+(1+c)a(z)\,, \end{aligned} \quad (5)$$

[8] K. Wilson, Ph.D. thesis, California Institute of Technology (unpublished).

[9] G. Wanders, Nuovo Cimento 23, 817 (1962).

[10] C. H. Albright and W. D. McGlinn, Nuovo Cimento 25, 193 (1962).

where $s(z)$ and $a(z)$ are symmetric and antisymmetric functions, respectively. Before proceeding we must discuss two "trivial" solutions, trivial in the sense that they possess only two Riemann sheets. The reader will recall that the solutions obtained by Wanders extend through an infinite number of Riemann sheets. The first such solution is obtained by setting $a(z)=0$. This solution, which is independent of the crossing parameter c, reduces to a single-channel problem with trivial crossing symmetry since the S-matrix elements are identical and symmetric. The general solution of unitarity in this case is known and is the function $D(z)$ discussed by Wanders.

The other solution is obtained by setting $s(z)=0$. In this case the unitarity condition can be satisfied only if $c=0$, and the resultant crossing matrix will be recognized as corresponding to the static-model problem solved by Castillejo, Dalitz, and Dyson.[11] The general form for an antisymmetric S-matrix element is known. Having discussed these special cases, we may carry out the factorization of the crossing solution [Eq. (5)]. This factorization is the key step in the procedure and leads to the expressions

$$\begin{aligned} S_1(z)&=A(z)[B(z)-(1-c)],\\ S_2(z)&=A(z)[B(z)+(1+c)], \end{aligned} \tag{6}$$

where $A(z)$ and $B(z)$ are antisymmetric, real analytic, meromorphic functions in the cut z plane.

The unitarity requirement on the S-matrix elements implies that the only branch point on the positive real axis for the functions $A(z)$ and $B(z)$ is at $z=1$, and that this branch point is of the type $(z-1)^{1/2}$. By Eq. (3) the continuations $A^{(2)}(z)$ and $B^{(2)}(z)$ on the second sheet of this branch point are given by

$$\begin{aligned} B^{(2)}(z)&=-B(z)-2c,\\ A^{(2)}(z)A(z)&=-\frac{1}{[B(z)-(1-c)][B(z)+(1+c)]}. \end{aligned} \tag{7}$$

Wanders has obtained the general solution for $B(z)$ for the case $2c=-1$.[12] It is a trivial matter to carry through his analysis for the general case and we will not repeat the details here. The result is[13]

$$B(z)=-(2c/\pi)\arcsin(z)+i(z^2-1)^{1/2}\beta(z), \tag{8}$$

where $\beta(z)$ is antisymmetric, real analytic, and meromorphic in the whole z plane. It will be convenient later to have the alternative expression

$$\arcsin(z)=\pi/2+i\ln[z+(z^2-1)^{1/2}]. \tag{9}$$

The difficult task now is to find solutions of Eq. (7) for $A(z)$. Wanders has shown that the general solution for $A(z)$ may be written

$$A(z)=A_0(z)D(z), \tag{10}$$

where $A_0(z)$ is any special solution of (7), and $D(z)$ is an arbitrary, symmetric "S-matrix element." That is, $D(z)$ is real analytic and meromorphic in the cut z plane and satisfies the second-sheet continuation properties of the $S_\alpha(z)$. The symmetry requirement then determines the general form for $D(z)$.[9]

To this point we have done nothing more than duplicate Wanders' procedure. Clearly, if unitarity and crossing relations are to lead to the prediction of symmetries in this model, then it must occur in the final step, the determination of special solutions for $A(z)$. For the crossing matrix he considered, Wanders found that the special solution for $A(z)$ could be expressed in terms of a rational function in $B(z)$. In fact, upon consideration of Eq. (7) one might conclude that all special solutions for $A(z)$ must be expressible in terms of rational functions in $B(z)$.

We pursue this conjecture by systematically analyzing the possible rational forms. The antisymmetry requirement demands that $B(z)$ occur as a linear factor in an otherwise symmetric form. Also, $A(z)\sim 1/B(z)$ for "very large" $B(z)$. Finally, the product $A^{(2)}(z)A(z)$ must reduce to the inverse of a quadratic form in $B(z)$ in order that it satisfy Eq. (7). We begin with the simplest possibility

$$A(z)=1/B(z). \tag{11}$$

It is readily verified that (11) is a special solution of (7) if and only if $c=\pm 1$. Similarly, the next simplest possibility

$$A(z)=\frac{B(z)}{[B(z)+2c][B(z)-2c]}$$

is a special solution if and only if $c=\pm\frac{1}{3}$, and the form

$$A(z)=\frac{[B(z)+2c][B(z)-2c]}{B(z)[B(z)+4c][B(z)-4c]}$$

is a solution if and only if $c=\pm\frac{1}{5}$.

This process may be extended in the obvious manner and leads to the following conclusion. Special solutions for $A(z)$ of the form found by Wanders, namely, rational functions in $B(z)$, exist only for values of the crossing-matrix parameter given by[14]

$$c=\pm 1/(2n+1),\quad n=0,1,2,3,\cdots. \tag{12}$$

[11] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[12] The crossing matrix for the problem solved by Wanders has $c=-\frac{1}{3}$. The fact that this equation implies $c=-\frac{1}{2}$ is due to a slightly different choice of normalization in the solution of the crossing relation, Eq. (5).

[13] The definition of the square root $(z^2-1)^{1/2}$ is an important factor in the mathematical analysis. We follow Wanders' choice, namely $\mathrm{Im}(z^2-1)^{1/2}\geq 0$ in the z-plane cut along $(-\infty,-1)$ and $(1,\infty)$; $[(z^2-1)^{1/2}]^*=-(z^{*2}-1)^{1/2}$. It follows that $i(z^2-1)^{1/2}$ is real analytic in the cut z plane.

[14] After this analysis was completed we were reminded that, as so frequently happens, Professor C. Goebel had studied the same problem and reached the same conclusion (unpublished). To our knowledge, Professor Goebel did not pursue the question of a general solution to the problem.

It is shown in Appendix II that these values of the parameter correspond to the crossing matrices for the scattering of particles with isotopic spin n by particles with isotopic spin $\frac{1}{2}$, or to the scattering of spinless particles by particles with spin $\frac{1}{2}$ in the orbital angular momentum state $l=n$. This last interpretation is meaningful, of course, only in the static limit since all angular-momentum partial-wave amplitudes are coupled through crossing in the relativistic case.

At any rate, if we can prove that all special solutions for $A(z)$ may be expressed as rational functions in $B(z)$, then we can reasonably argue that the requirements of unitarity and crossing symmetry should be expected to lead to the prediction of symmetries among the strongly interacting particles. We must therefore confront the problem of determining special solutions for $A(z)$ for arbitrary values of the parameter c. This mathematical question is treated in the following section. We will return to the matter of the interpretation of the above results in the concluding section.

III. GENERAL SOLUTION

There is a rule of thumb in strong-interaction calculations that if it is not difficult to satisfy the unitarity requirement in a given formalism, then it will be very difficult to satisfy crossing symmetry in that formalism, and vice versa. For example, the Mandelstam representation[15] incorporates crossing symmetry with ease, but satisfying the unitarity condition is a very difficult problem. In the partial-wave dispersion relations, on the other hand, it is much easier to satisfy unitarity than it is to guarantee the crossing relations.

The present problem, finding the general solution for $A(z)$ in the two-channel static model with an arbitrary crossing matrix, is essentially a problem of unitarity; the crossing relations have already been solved. It follows from the opening remarks of this section that we should employ a formalism in which the unitarity condition may be trivially satisfied. Such a formalism is provided by the "phase-shift dispersion relations."[10] The basic simplification in this case is that the unitarity requirement on the function

$$\Delta_\alpha(z)=\ln S_\alpha(z) \tag{13}$$

is a linear condition. The price one pays for this simplification is the introduction of additional branch points at the zeros and poles of the S-matrix element. In the present case, however, this complication is inessential and the general solution can be obtained.

We utilize the fact that any solution for $A(z)$ can be written in the form of Eq. (10) where $A_0(z)$ is antisymmetric, real analytic, and meromorphic in the cut z plane, and $D(z)$ is symmetric, real analytic, and meromorphic in the cut z plane. The reader will easily convince himself that given any special solution $A_0(z)$ of Eq. (7), there exists a $D(z)$ such that the product $A(z)=A_0(z)D(z)$ is devoid of zeros and poles (away from the two branch cuts) except for a simple pole at the origin. In other words, there exists a special solution of the form

$$A(z)=C(z)/z\,, \tag{14}$$

where $C(z)$ is a symmetric, real analytic, nonvanishing entire function in the cut z plane.

In terms of this special solution the S-matrix elements [Eq. (6)] are given by

$$\begin{aligned}S_1(z)&=C(z)[B(z)-(1-c)]/z\,,\\ S_2(z)&=C(z)[B(z)+(1+c)]/z\,.\end{aligned} \tag{15}$$

These S-matrix elements, which differ from the general solution by only the common factor of an arbitrary $D(z)$, must satisfy the unitarity condition on the physical branch cut. For the "phase shifts" $\Delta_\alpha(z)$ [Eq. (13)], which by virtue of (15) are written

$$\begin{aligned}\Delta_1(z)&=\ln C(z)+\ln[B(z)-(1-c)]-\ln(z)\,,\\ \Delta_2(z)&=\ln C(z)+\ln[B(z)+(1+c)]-\ln(z)\,,\end{aligned} \tag{16}$$

the unitarity condition leads to the equation for the discontinuity across the physical cut[10]

$$\Delta_\alpha(z)-\Delta_\alpha^{(2)}(z)=2\Delta_\alpha(z)\,,\quad z\geqslant 1\,. \tag{17}$$

It is understood that the branch cuts arising from zeros or poles of the S-matrix elements are to be drawn away from the physical cut in such a way as to preserve the real analyticity of the $\Delta_\alpha(z)$. This is always possible for real analytic $S_\alpha(z)$. In particular, the cut associated with the term $\ln(z)$ in (16) is chosen to be $(-\infty, 0)$.

It is now a matter of a little algebra to show that (16) and (17), for each of the phase shifts separately, lead to the expression on the physical cut

$$\ln C(z)+\ln C^{(2)}(z)=2\ln(z)-\ln\{-[B(z)-(1-c)][B(z)+(1+c)]\}\,. \tag{18}$$

It is worthwhile to note that for $z\geqslant 1$, using $B^{(2)}(z)=B^*(z)$ and Eqs. (8) and (9), we may write

$$\begin{aligned}-[B(z)-(1-c)][B(z)+(1+c)]&=|B(z)-(1-c)|^2\\ &=|B(z)+(1+c)|^2=1+[F(z)]^2\,,\end{aligned} \tag{19}$$

where $F(z)$ is real for $z\geqslant 1$ and is given by

$$F(z)=(2c/\pi)\ln[z+(z^2-1)^{1/2}]-(z^2-1)^{1/2}\beta(z)\,. \tag{20}$$

In (19) and (20) the functions are defined in the limit as the physical cut is approached from above. It follows that the argument of the last logarithmic term on the right-hand side of (18) is greater than or equal to unity.

We may now employ a standard mathematical tech-

[15] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); **115**, 1741 and 1752 (1959).

nique to determine $C(z)$. The function

$$\chi(z)=\ln C(z)/i(z^2-1)^{1/2} \tag{21}$$

is symmetric, real analytic, and entire in the z-plane cut along $(-\infty, -1)$ and $(1, \infty)$. The discontinuity of this function across the right-hand cut (and by symmetry across the left-hand cut) is

$$\chi(z)-\chi^{(2)}(z)=[\ln C(z)+\ln C^{(2)}(z)]/i(z^2-1)^{1/2}. \tag{22}$$

Using (18) we obtain the general form

$$\chi(z)=-\frac{1}{\pi}\int_1^\infty \frac{2\ln(x)-\ln\{-[B(x)-(1-c)][B(x)+(1+c)]\}}{(x^2-1)^{1/2}(x^2-z^2)}xdx+\Phi(z), \tag{23}$$

where $\Phi(z)$ is a real analytic, symmetric entire function. Part of the integral in (23) may be evaluated according to

$$-\frac{1}{\pi}\int_1^\infty \frac{2\ln(x)xdx}{(x^2-1)^{1/2}(x^2-z^2)}=\frac{\ln[1-i(z^2-1)^{1/2}]}{i(z^2-1)^{1/2}}. \tag{24}$$

Combining (21), (23), and (24), we obtain the general form for $C(z)$

$$C(z)=[1-i(z^2-1)^{1/2}]\exp[i(z^2-1)^{1/2}\Phi(z)]\times\exp\left[\frac{i(z^2-1)^{1/2}}{\pi}\int_1^\infty \frac{\ln\{-[B(x)-(1-c)][B(x)+(1+c)]\}}{(x^2-1)^{1/2}(x^2-z^2)}xdx\right]. \tag{25}$$

The exponential factor involving $\Phi(z)$ in Eq. (25) is nothing more than a symmetric S-matrix element and may be included in the arbitrary $D(z)$ factor in the general solution for $A(z)$. Finally, using (14), (19), (20), and (25), we may write the special solution for $A(z)$ for any value of the crossing-matrix parameter in the form

$$A(z)=\frac{[1-i(z^2-1)^{1/2}]}{z}\times\exp\left[\frac{i(z^2-1)^{1/2}}{\pi}\int_1^\infty \frac{\ln[1+F^2(x)]xdx}{(x^2-1)^{1/2}(x^2-z^2)}\right]. \tag{26}$$

This special solution fulfills the requirements imposed upon $A(z)$. As one check, note that on the physical cut

$$|A(z)|=[1+F^2(z)]^{-1/2}, \tag{27}$$

which by Eqs. (6) and (19) is the necessary condition for the physical S-matrix elements to have unit magnitude. One particularly interesting feature of this special solution [Eq. (26)] is its seemingly trivial dependence on the crossing-matrix parameter c. This parameter enters only as a linear factor in one of the terms comprising the function $F(z)$ [Eq. (20)]. From this point of view it is difficult to imagine a criterion that would select certain values of c as being more significant than the others.

In concluding this section we note that the insight provided by the special solutions obtained in Sec. II, it is possible to evaluate the integral in (26) for those values of c given by (12). The results agree, as they must, with the "rational" solutions within a factor of $D(z)$. For other values of c we are unable to carry out the integral in (26) in an analytic fashion.

IV. CONCLUSIONS

We have considered a model for the strong interactions that incorporates the unitarity condition and a nontrivial crossing relation. The model is an extension of the two-channel static model in that it permits the introduction of arbitrary crossing matrices of a class suitable to the static model. Prior to this work, exact solutions of the unitarity and crossing requirements had been obtained by Wilson[8] and by Wanders[9] for a specific choice of the crossing matrix. A striking feature of their solutions is that they are largely independent of a number of the dynamical assumptions that go into the static model. For this reason one might hope that certain aspects of the exact solutions will hold as well in more realistic models for the strong interactions.

With this possibility in mind, we have asked whether the requirement that exact solutions exist, within the framework of the static model, leads to restrictions on the crossing matrices and therefore to the prediction of symmetries. We are forced to answer this question in the negative as we have exhibited solutions for all values of the crossing-matrix parameter. Nevertheless, it is an intriguing fact that exact solutions of a particularly simple form, namely rational functions in $B(z)$, exist only for crossing matrices corresponding to the symmetry group SU(2). Have we overlooked a general criterion that distinguishes these solutions from the solutions for arbitrary values of c?

One possibility concerns the asymptotic behavior of the solutions. Perhaps the solution for arbitrary c [Eq. (26)] is usually characterized by an essential singularity at infinity; the "rational" solutions clearly do not have this behavior. The answer to this question is again in the negative. In fact, with the freedom of choice provided by the functions $D(z)$ and $\beta(z)$, it seems to us unlikely that any general criterion could select certain values of c as preferred. There is another possibility, however, that is more in keeping with the bootstrap hypothesis. It may be that unitarity and crossing symmetry are not enough; that specific dynamical requirements such as a minimum number of zeros in the S-matrix elements or, perhaps, analyticity in the coupling constants must also be imposed.[10,16] We have not pursued these questions although we think they are of considerable interest.

Instead, let us suppose that a criterion is found which admits only the "rational" solutions for the S-matrix elements. How do we interpret the resulting prediction of symmetry? It is well known that the static model is effectively invariant under the "interchange" of spin and isospin [the threshold conditions on the partial-wave phase shifts can easily be incorporated in the solution of Eq. (26) by means of subtractions]. Yet, while few people would object to the "bootstrapping" of an internal symmetry such as isotopic spin, the prediction of angular-momentum symmetry is an entirely different matter; the isotropy of space hardly seems related to unitarity or the existence of crossing relations. The answer clearly lies in the limitations of the static model. The crossing matrices relating isotopic-spin amplitudes remain unchanged in going to the relativistic case, while those relating angular-momentum amplitudes immediately become infinite in dimension. It follows that the only interpretation which is likely to persist in a relativistic model is that concerned with internal symmetries.

There is another, more serious difficulty in the interpretation of the symmetry "predicted" by the rational solutions. Let us confine our attention to the isotopic-spin case. It is evident that one of the particles involved in the scattering process must have isospin $\frac{1}{2}$; otherwise the number of total isotopic-spin amplitudes, and hence the dimensionality of the crossing matrix, would exceed two. We pointed out in Sec. II that the rational solutions correspond to the scattering of particles with isospin $\frac{1}{2}$ by particles with *integer* isotopic spin. What about particles with half-odd-integer isospin? The crossing matrix relating isospin amplitudes for N-K scattering with those for N-$\bar{K}$ scattering, as an example, is given by Eq. (4) with $c=-\frac{1}{2}$.[17] The static model for N-K scattering (in the s-wave amplitude, say, to avoid enlarging the crossing matrix) is therefore not characterized by rational solutions for the S-matrix elements. From this point of view it is perhaps reassuring that exact solutions exist for all values of the crossing-matrix parameter.

We are hopeful that the techniques discussed in this paper will permit the solution of static-model problems with crossing matrices of higher dimensionality. It would be very satisfying, for example, to know the exact solution to the Low equation[18] for pion-nucleon scattering (in the one-meson approximation). On a more fundamental level is the question of the extension of these ideas to the relativistic case. A relativistic model will, of course, pose a much more difficult problem; it should incorporate the fact that crossing symmetry relates three physical processes (not just two as in the static model) and, as a consequence, it would seem to be a problem in at least two complex variables. Yet until such questions are answered, we will not know whether the symmetries evinced by the strong interactions are consequences simply of the requirements of crossing symmetry and unitarity, whether they are due to more stringent dynamical requirements such as the bootstrap hypothesis, or whether they are "just there."

[16] W. D. McGlinn and C. H. Albright, Nuovo Cimento **27**, 834 (1963); R. H. Capps, Phys. Rev. **128**, 2842 (1962).

[17] See, for example, R. H. Capps, Phys. Rev. **121**, 291 (1961); B. W. Lee, *ibid.* **125**, 2201 (1962).

[18] F. E. Low, Phys. Rev. **97**, 1392 (1955).

APPENDIX I: STATIC MODEL CROSSING MATRICES

The purpose of this Appendix is to derive two general properties of the crossing matrices appropriate to the static model. The results are not limited to the static model as will be clear from the development. On the other hand, the results are not applicable to all crossing matrices encountered in strong interaction calculations. The basic limitation is that the two physical processes related by crossing symmetry must be represented by amplitudes of the same general structure. For example, the results apply to pion-nucleon scattering in the case where the crossed channel is also pion-nucleon scattering but not where the crossed channel is pion-pion going to nucleon-antinucleon.

We employ a somewhat abbreviated notation for simplicity. Let the crossing relation between two physical processes be summarized in the form

$$M_{cb}(s_i')=M_{bc}(s_i)\,, \tag{A1}$$

where the subscripts b and c represent the quantum numbers of the appropriate initial and final states and the s_i are the appropriate kinematical variables. Consider the expansion of these amplitudes in terms of a complete set of eigenamplitudes corresponding to different values of some conserved quantity

$$\begin{aligned} M_{bc}(s_i)&=\textstyle\sum_\gamma h_\gamma(s_i)P_{bc}{}^\gamma\,,\\ M_{cb}(s_i')&=\textstyle\sum_{\gamma'} h_{\gamma'}(s_i')P_{cb}{}^{\gamma'}\,, \end{aligned} \tag{A2}$$

where the $P_{bc}{}^\gamma$ are projection operators for the conserved quantity.

The projection operators enjoy the properties

$$\textstyle\sum_\gamma P_{bc}{}^\gamma=\delta_{bc}\,, \tag{A3}$$

$$\textstyle\sum_d P_{bd}{}^\gamma P_{dc}{}^{\gamma'}=\delta_{\gamma\gamma'}P_{bc}{}^\gamma\,, \tag{A4}$$

and, for the type of crossing symmetry under consideration, are related by

$$P_{bc}{}^{\gamma}=\sum_{\gamma'} A_{\gamma'\gamma}P_{cb}{}^{\gamma'}. \tag{A5}$$

Note the "backwards" order of the subscripts on the crossing matrix $A_{\gamma'\gamma}$. The first property of the crossing matrix follows from applying Eq. (A5) twice

$$P_{bc}{}^{\gamma}=\sum_{\gamma'} A_{\gamma'\gamma}P_{cb}{}^{\gamma'}=\sum_{\gamma'} A_{\gamma'\gamma}\sum_{\gamma''} A_{\gamma''\gamma'}P_{bc}{}^{\gamma''},$$

which leads to

$$\sum_{\gamma'} A_{\gamma''\gamma'}A_{\gamma'\gamma}=\delta_{\gamma''\gamma'}.$$

The crossing matrix is therefore its own inverse. The second property follows from the completeness condition [Eq. (A3)]

$$\delta_{bc}=\sum_{\gamma} P_{bc}{}^{\gamma}=\sum_{\gamma,\gamma'} A_{\gamma'\gamma}P_{cb}{}^{\gamma'}=\sum_{\gamma'} P_{cb}{}^{\gamma'},$$

which leads to (with a certain lack of rigor)

$$\sum_{\gamma} A_{\gamma'\gamma}=1.$$

The sum over each row of the crossing matrix equals unity.

The crossing relation for the eigenamplitudes is obtained by combining (A1), (A2), and (A5) in the form

$$\sum_{\gamma} h_{\gamma}(s_i)P_{bc}{}^{\gamma}=\sum_{\gamma'} h_{\gamma'}(s_i')P_{cb}{}^{\gamma'}=\sum_{\gamma',\gamma''} h_{\gamma'}(s_i')A_{\gamma''\gamma'}P_{bc}{}^{\gamma''}.$$

Use of the orthonormality property [Eq. (A4)] leads to the desired crossing relation

$$h_{\gamma}(s_i)=\sum_{\gamma'} A_{\gamma\gamma'}h_{\gamma'}(s_i').$$

It is trivial to see that these properties of the crossing matrix must hold in the static model. Equation (2) requires that the crossing matrix equal its inverse, and the condition on the sum over each row of the matrix follows from the requirement that the crossing relation hold for both the S-matrix elements and the associated transition amplitudes.

APPENDIX II: SU(2) SUBSET OF CROSSING MATRICES

In this Appendix we derive a special subset of two-by-two crossing matrices corresponding to the SU(2) group. While the physical interpretation of this subset follows more easily in the language of isotopic spin, it is convenient to employ the formalism of angular momentum since the mathematics is more familiar. With regard to isotopic spin, this subset of crossing matrices applies to the scattering of isospin-$\frac{1}{2}$ particles by particles with integer (but not half-odd-integer) isospin. In the angular-momentum case it represents the scattering of a spinless particle by a particle with spin $\frac{1}{2}$ in the orbital angular-momentum state l. This last interpretation applies, of course, only in the static limit since we are restricting our attention to two-by-two matrices.

To derive the result we use the angular-momentum projection operators for spin-0–spin-$\frac{1}{2}$ scattering. These operators, defined in the barycentric system, are relativistically correct; they are not limited to the static model. In fact, they provide a particularly simple way to generate the partial-wave expansion for baryon-meson scattering.[19] We write

$$\begin{aligned} Q_{l-}(2,1)&=-P_{l-1}'(x)+\boldsymbol{\sigma}\cdot\mathbf{q}_2\boldsymbol{\sigma}\cdot\mathbf{q}_1P_l'(x),\\ Q_{l+}(2,1)&=P_{l+1}'(x)-\boldsymbol{\sigma}\cdot\mathbf{q}_2\boldsymbol{\sigma}\cdot\mathbf{q}_1P_l'(x),\end{aligned} \tag{B1}$$

where $\mathbf{q}_1$ ($\mathbf{q}_2$) is the unit vector for the initial (final) three momentum in the barycentric system, the $\boldsymbol{\sigma}$ are the familiar Pauli matrices, the $P_l(x)$ are the Legendre polynomials, and x is given by $x=\mathbf{q}_1\cdot\mathbf{q}_2$. The subscripts on the projection operators Q represent the usual partial-wave notation in which $l\pm$ stands for $J=l\pm\frac{1}{2}$, where J is the total angular momentum.

The completeness property for these projection operators takes the form

$$Q_{l+}(2,1)+Q_{l-}(2,1)=P_{l+1}'(x)-P_{l-1}'(x)=(2l+1)P_l(x), \tag{B2}$$

which represents the statement that the sum of both total-angular-momentum projection operators is just the projection operator for the related orbital angular momentum. The orthonormality property of the projection operators is given by

$$\begin{aligned}\sum_n Q_{l\pm}(2,n)Q_{l'\pm}(n,1)&=\delta_{ll'}Q_{l\pm}(2,1),\\ \sum_n Q_{l\pm}(2,n)Q_{l'\mp}(n,1)&=0,\end{aligned} \tag{B3}$$

where the sum over intermediate states means

$$\sum_n=\sum_{\text{spins}}\int d\Omega_n/4\pi.$$

The elements of the crossing matrix of interest satisfy the condition

$$\begin{pmatrix}Q_{l-}(1,2)\\ Q_{l+}(1,2)\end{pmatrix}=\begin{pmatrix}a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}\begin{pmatrix}Q_{l-}(2,1)\\ Q_{l+}(2,1)\end{pmatrix}. \tag{B4}$$

Using the familiar properties of the Pauli spin matrices, we write

$$\begin{aligned}Q_{l-}(1,2)&=-P_{l-1}'(x)+\boldsymbol{\sigma}\cdot\mathbf{q}_1\boldsymbol{\sigma}\cdot\mathbf{q}_2P_l'(x)\\ &=-P_{l-1}'(x)+2xP_l'(x)-\boldsymbol{\sigma}\cdot\mathbf{q}_2\boldsymbol{\sigma}\cdot\mathbf{q}_1P_l'(x),\\ Q_{l+}(1,2)&=P_{l+1}'(x)-\boldsymbol{\sigma}\cdot\mathbf{q}_1\boldsymbol{\sigma}\cdot\mathbf{q}_2P_l'(x)\\ &=P_{l+1}'(x)-2xP_l'(x)+\boldsymbol{\sigma}\cdot\mathbf{q}_2\boldsymbol{\sigma}\cdot\mathbf{q}_1P_l'(x).\end{aligned} \tag{B5}$$

Inserting (B5) into (B4) and using the recurrence relations for the Legendre polynomials, we obtain the crossing matrix for a given orbital angular momentum l

$$A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}=\frac{1}{2l+1}\begin{pmatrix}-1 & 2l+2\\ 2l & 1\end{pmatrix}. \tag{B6}$$

This crossing matrix clearly satisfies the properties de-

[19] See, for example, S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

scribed in Appendix I. Since l can be any non-negative integer, these crossing matrices correspond precisely to the "rational" solutions [Eq. (12)] for the S-matrix elements. Note, finally, that this derivation holds for $l=0$, even though one channel is a "nonsense" channel, because the recurrence relations between the Legendre polynomials may be formally extended to the $l=0$ case.

Note added in proof. After this article was submitted for publication, Professor K. Wilson directed our attention to a paper by J. Rothleitner [J. Rothleitner, Z. Physik **177**, 287 (1964)] in which general solution for the two-channel static model is obtained by a technique substantially different from our own. We wish to thank Professor Wilson for informing us of Rothleitner's interesting work.

PYSICAL REVIEW VOLUME 136, NUMBER 5B 7 DECEMBER 1964

Leptonic Decay of Hyperons in an Intermediate-Boson Theory

SEITARO NAKAMURA
Department of Physics, University of Tokyo and Nihon University, Tokyo, Japan

AND

KUNIO ITAMI
Department of Physics, University of Tokyo, Tokyo, Japan

(Received 11 May 1964)

The energy spectrum of electrons emitted in the beta decay of hyperons is calculated on the basis of the intermediate-boson theory of Tanikawa and Watanabe. In the bare-nucleon approximation, an appreciable deviation of the spectrum shape from that calculated on the local limit of the universal Fermi interaction is expected.

RECENTLY, Marshak *et al.*[1] proposed a scheme of the weak interactions in which the weak boson responsible for the leptonic decay of hyperons is different from that responsible for the β decay of nucleon. On the other hand, Sato and one of the authors (SN)[2] considered a scheme in which the weak boson of the Tanikawa type[3,4] (with the baryon number) is responsible for the leptonic decay of hyperons, while both the weak boson of the Yukawa type (without baryon number) and that of the Tanikawa type take part in the β decay of nucleon. These two schemes are alike in distinguishing the leptonic decay of hyperons from the β-decay of nucleon by the difference in the weak-boson channel. We shall show that the expected energy spectrum of the electron emitted in the leptonic decay of hyperons should differ appreciably in these two schemes, which could therefore be tested by precision measurements.

For the sake of simplicity let us calculate the energy spectrum of the electrons emitted in the β decay of the Λ hyperon when it is mediated by the Tanikawa boson by using the lowest-order perturbation. We shall limit ourselves to the case in which the Hamiltonian for the interaction among leptons, baryons, and boson leads to the $V\pm A$ coupling types

$$\bar{p}\gamma_\alpha(1\pm\gamma_5)\Lambda\cdot\bar{e}\gamma_\alpha(1\pm\gamma_5)\nu \quad \text{and} \quad \bar{p}\gamma_\alpha(1\mp\gamma_5)\Lambda\cdot\bar{e}\gamma_\alpha(1\pm\gamma_5)\nu$$

in the local limit.

(i) Spin-0 boson

(a) $V+A$

$$H_0=[f_{\Lambda 0}\bar{\Lambda}(1\pm\gamma_5)e+g_{\Lambda 0}\bar{p}(1\pm\gamma_5)\nu]\phi_\Lambda+\text{H.c.} \quad (1a)$$

(b) $V-A$

$$H_0'=[f_{\Lambda 0}'\bar{p}(1\pm\gamma_5)e^c+g_{\Lambda 0}'\bar{\Lambda}(1\pm\gamma_5)\nu^c]\phi_\Lambda'+\text{H.c.} \quad (1b)$$

(ii) Spin-1 boson

(a) $V-A$

$$H_1=i[f_{\Lambda 1}\bar{\Lambda}\gamma_\alpha(1\pm\gamma_5)e+g_{\Lambda 1}\bar{p}\gamma_\alpha(1\pm\gamma_5)\nu]\phi_{\Lambda\alpha}+\text{H.c.} \quad (2a)$$

(b) $V+A$

$$H_1'=i[f_{\Lambda 1}'\bar{p}\gamma_\alpha(1\pm\gamma_5)e^c+g_{\Lambda 1}'\bar{\Lambda}\gamma_\alpha(1\pm\gamma_5)\nu^c]\phi_{\Lambda\alpha}'+\text{H.c.} \quad (2b)$$

Here e and ν denote the annihilation operators of electron and neutrino, respectively. We assume that ν and its charge conjugation ν^c are described by the four-component Dirac spinors.

[1] R. E. Marhsak, C. Ryan, T. K. Radha, and K. Raman, Phys. Rev. Letters **11**, 396 (1963); Nuovo Cimento **16**, 408 (1964).
[2] S. Nakamura and S. Sato, Progr. Theoret. Phys. (Kyoto) **29**, 325 (1963).
[3] Y. Tanikawa, Progr. Theoret. Phys. (Kyoto) **3**, 338 (1948); Proc. Intern. Conf. Theoret. Phys. Kyoto Tokyo, Japan, 1953, 369 (1954); Progr. Theoret. Phys. Kyoto, **10**, 361 (1953); Y. Tanikawa and K. Saeki, Progr. Theoret. Phys. (Kyoto) **10**, 232 (1953); Y. Tanikawa, Phys. Rev. **108**, 1615 (1957); Y. Tanikawa and S. Watanabe, Phys. Rev. **113**, 1344 (1959).
[4] S. Nakamura and K. Itami, Progr. Theoret. Phys. (Kyoto) **26**, 274 (1961).